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A note on scattering theory in non-relativistic quantum electrodynamics

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Abstract. Scattering theory of photons by a free electron is considered rigorously in a cut-off non-relativistic quantum electrodynamics and within the dipole approximation. The dipole approximation permits one to obtain the dressed one-electron states as well as the photon scattering cross section exactly. It is noted that the cross section of the Thomson scattering is obtained in the low photon-energy limit only after the electron mass is renormalised.

1. Introduction

In this paper we give a rigorous theory of scattering of photons by a free electron in a cut-off non-relativistic quantum electrodynamics and within the dipole approximation. Apart from the mathematical technicalities, the main point is to show that the mass renormalisation of the electron is necessary to obtain the cross section of the Thomson scattering in the low photon-energy limit.

The system we consider consists of one non-relativistic free electron interacting with a quantised radiation field. As usual, we use the Coulomb gauge. We introduce an ultraviolet cut-off which makes the interaction a well defined self-adjoint operator in the Fock space.

The model, without the $\boldsymbol{A}^2$ term of the interaction, was considered rigorously by Blanchard (1969) with special attention to the problem of infrared catastrophe. We keep the term, however, because it is essential for the scattering of photons by the electron. In a previous paper (Arai 1981a), a more general framework of the model was given including the case in which the electron is bound in an external potential, and the self-adjointness and the basic spectral properties of the Hamiltonians were established.

The infrared problem arises in our model as in the cases of the models of scalar electrons interacting with massless bosons (Fröhlich 1973) and the model by Blanchard (1969). This problem, however, can be overcome in the same way as by Fröhlich (1973). Namely, we first introduce an infrared cut-off in the interaction in addition to the ultraviolet cut-off and prove the existence of dressed one-electron states (DES) of the total Hamiltonian. We remark that, without the infrared cut-off, the total Hamiltonian is well defined and self-adjoint in the Fock space, but the DES does not exist; this is the infrared catastrophe in our model. The infrared cut-off can be removed after the Wightman distributions are constructed as the expectation values of products of the radiation field with respect to the DES.

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The outline of this paper is as follows: § 2 defines the notation and gives some basic facts. In § 3 the existence and uniqueness (respectively absence) of the DES is proved and the spectral analysis carried out for the total Hamiltonian with (respectively without) infrared cut-off. In § 4 we remove the infrared cut-off in terms of the Wightman distributions. Then, by the reconstruction theorem, we obtain a theory without infrared cut-off, but having the DES. Section 5 is devoted to the construction of the scattering theory. We obtain the exact form of the total cross section for the scattering of a photon by the electron, by which the statement with regard to the mass renormalisation mentioned above is shown.

2. Notation and basic facts

We consider a system of one non-relativistic electron interacting with a quantised radiation field. We use the unit system in which $c = \hbar = 1$.

The Hilbert space $\mathscr{H}$ of state vectors for the system is given by

$$\mathscr{H} = L^2(R^3) \otimes \mathscr{F}^{\mathrm{EM}}, \tag{2.1}$$

where $\mathscr{F}^{\mathrm{EM}}$ is the Fock space for photons in the Coulomb gauge.

Let $a^{(r)}(f)$ and $a^{(r)*}(f)$, $f \in L^2(R^3)$, $r = 1, 2$, be the smeared annihilation and creation operators in $\mathscr{F}^{\mathrm{EM}}$ for a photon with polarisation vector $\boldsymbol{e}^{(r)}(\boldsymbol{k}) \in R^3$, $r = 1, 2$, relative to its momentum $\boldsymbol{k}$. Then, in the Coulomb gauge, the time-zero radiation field $\boldsymbol{A}(f)$ and its canonical conjugate $\boldsymbol{\pi}(f)$ are given by

$$A_\mu(f) = \frac{1}{\sqrt{2}} \sum_{r=1}^{2} [a^{(r)*}(\hat{f} e_\mu^{(r)}/\sqrt{\omega}) + a^{(r)}(\tilde{\hat{f}} e_\mu^{(r)}/\sqrt{\omega})] \qquad \hat{f}/\sqrt{\omega} \in L^2(R^3) \tag{2.2}$$

$$\mu = 1, 2, 3$$

$$\pi_\mu(f) = \frac{\mathrm{i}}{\sqrt{2}} \sum_{r=1}^{2} [a^{(r)*}(\sqrt{\omega} \hat{f} e_\mu^{(r)}) - a^{(r)}(\sqrt{\omega} \tilde{\hat{f}} e_\mu^{(r)})] \qquad \sqrt{\omega} \hat{f} \in L^2(R^3) \tag{2.3}$$

where $\hat{f}$ denotes the Fourier transform of f and

$$\tilde{\hat{f}}(\boldsymbol{k}) = \hat{f}(-\boldsymbol{k}) \tag{2.4}$$

$$\omega(\boldsymbol{k}) = |\boldsymbol{k}|. \tag{2.5}$$

The free Hamiltonian of photons H_0^{EM} is a non-negative self-adjoint operator in $\mathscr{F}^{\mathrm{EM}}$ and is written symbolically as

$$H_0^{\mathrm{EM}} = \sum_{r=1}^{2} \int \mathrm{d}^3\boldsymbol{k}\, \omega(\boldsymbol{k}) a^{(r)*}(\boldsymbol{k}) a^{(r)}(\boldsymbol{k}) \tag{2.6}$$

with $a^{(r)}(\boldsymbol{k})(a^{(r)*}(\boldsymbol{k}))$ being the symbolic annihilation (creation) operator as given by

$$a^{(r)}(f) = \int \mathrm{d}^3\boldsymbol{k}\, a^{(r)}(\boldsymbol{k}) f(\boldsymbol{k}). \tag{2.7}$$

Let $m > 0$ and $-e < 0$ be the physical mass and charge of the electron respectively. With minimal interaction between the electron and the radiation field, the total Hamiltonian in the dipole approximation is given formally by

$$H(K) = \frac{1}{2(m - \delta m(K))} :[(-\mathrm{i}\nabla) \otimes I + eI \otimes \boldsymbol{A}(\rho_K)]^2: + I \otimes H_0^{\mathrm{EM}}. \tag{2.8}$$

Here : : denotes the Wick ordering and ρ a real-valued function whose Fourier transform serves as an ultraviolet cut-off. The parameter $K \geq 0$ is the infrared cut-off. The function ρ_K is defined by

$$\hat{\rho}_K(\boldsymbol{k}) = \begin{cases} \hat{\rho}(\boldsymbol{k}) & |\boldsymbol{k}| \geq K \\ 0 & |\boldsymbol{k}| < K. \end{cases} \tag{2.9}$$

Throughout the present paper we assume that $\hat{\rho}$ is a rotation-invariant function, satisfying

$$\hat{\rho} > 0 \qquad \hat{\rho} \in \mathscr{S}(R^3) \qquad (2\pi)^{3/2}\hat{\rho}(\mathbf{0}) = \int \mathrm{d}^3\boldsymbol{x}\, \rho(\boldsymbol{x}) = 1. \tag{2.10}$$

The mass renormalisation $\delta m(K)$ is given by

$$\delta m(K) = \tfrac{2}{3}e^2 \|\hat{\rho}_K/\omega\|_0^2 \tag{2.11}$$

where $\| \; \|_0$ denotes the $L^2(R^3)$ norm.

Theorem 2.1. Let $m \neq \delta m(K)$. Then $H(K)$ is essentially self-adjoint on $D = D(\Delta \otimes I) \cap D(I \otimes H_0^{\mathrm{EM}})$. In particular, if $m > 2\delta m(K)$, then $H(K)$ is self-adjoint with $D(H(K)) = D$. Furthermore, if $m > \delta m(K)$(respectively $m < \delta m(K)$), then $H(K)$ is bounded below (respectively not bounded below).

For the proof, see Arai (1981a).

Since the electron momentum $(-\mathrm{i}\nabla)$ commutes with $H(K)$, it is conserved, implying a lack of electron recoil in our model, so that we can decompose $\mathscr{H}$ and $H(K)$ on the spectrum of $(-\mathrm{i}\nabla)$:

$$\mathscr{H} \simeq \int_{R^3}^{\oplus} \mathrm{d}^3\boldsymbol{p}\, \mathscr{H}(\boldsymbol{p}) \qquad \mathscr{H}(\boldsymbol{p}) = \mathscr{F}^{\mathrm{EM}} \qquad \boldsymbol{p} \in R^3 \tag{2.12}$$

$$H(K) \simeq \int_{R^3}^{\oplus} \mathrm{d}^3\boldsymbol{p}\, H(K, \boldsymbol{p}) \tag{2.13}$$

where

$$H(K, \boldsymbol{p}) = \frac{1}{2(m - \delta m(K))} : (\boldsymbol{p} + e\boldsymbol{A}(\rho_K))^2 : + H_0^{\mathrm{EM}}. \tag{2.14}$$

Theorem 2.2. Let $m \neq \delta m(K)$. Then, for all $\boldsymbol{p} \in R^3$, $H(K, \boldsymbol{p})$ is essentially self-adjoint on $D(H_0^{\mathrm{EM}})$. In particular, if $m > \delta m(K)$, then $H(K, \boldsymbol{p})$ is self-adjoint with $D(H(K, \boldsymbol{p})) = D(H_0^{\mathrm{EM}})$ and is bounded below.

Proof. The first half of the theorem can be proved in the same way as in theorem 2.1. We now prove the second half. We write $H(K, \boldsymbol{p})$ as

$$H(K, \boldsymbol{p}) = H_0^{\mathrm{EM}} + H_I^{(1)}(K, \boldsymbol{p}) + H_I^{(2)}(K) + \boldsymbol{p}^2/2(m - \delta m(K)) \tag{2.15}$$

where

$$H_I^{(1)}(K, \boldsymbol{p}) = \frac{e}{m - \delta m(K)} \boldsymbol{p} \cdot \boldsymbol{A}(\rho_K) \qquad H_I^{(2)}(K) = \frac{e^2}{2(m - \delta m(K))} : \boldsymbol{A}(\rho_K)^2 :. \tag{2.16}$$

By the basic estimates

$$\|a^{(r)}(f)\Psi\| \leq \|f/\sqrt{\omega}\|_0 \|(H_0^{EM})^{1/2}\Psi\| \tag{2.17}$$

$$\|a^{(r)*}(f)\Psi\| \leq \|f/\sqrt{\omega}\|_0 \|(H_0^{EM})^{1/2}\Psi\| + \|f\|_0\|\Psi\|$$

$$r = 1, 2 \qquad f, f/\sqrt{\omega} \in L^2(R^3) \qquad \Psi \in D((H_0^{EM})^{1/2}) \tag{2.18}$$

we have

$$\|H_I^{(1)}(K, \boldsymbol{p})\Psi\| \leq \varepsilon c_1(\boldsymbol{p})\|H_0^{EM}\Psi\| + d_1(\varepsilon, \boldsymbol{p})\|\Psi\| \tag{2.19}$$

$$\|H_I^{(2)}(K)\Psi\| \leq c\|H_0^{EM}\Psi\| + d\|\Psi\| \tag{2.20}$$

for all $\Psi \in D(H_0^{EM})$ with $\varepsilon > 0$ arbitrary and with c-numbers $c_1(\boldsymbol{p})$, $d_1(\varepsilon, \boldsymbol{p})$, c and d independent of K. Furthermore, we have from the positivity of H_0^{EM} and the canonical commutation relations

$$\operatorname{Re}(H_0^{EM}\Psi, H_I^{(2)}(K)\Psi) \geq -\frac{e^2}{2(m - \delta m(K))}(\|\hat{\rho}_K\|_0^2\|\Psi\|^2 + \|\hat{\rho}_K/\sqrt{\omega}\|_0^2(H_0^{EM}\Psi, \Psi))$$

$$\Psi \in D(H_0^{EM}). \tag{2.21}$$

By (2.19) and (2.21) we get

$$c_3(\boldsymbol{p})(\|H_0^{EM}\Psi\|^2 + \|(H_I^{(1)}(K, \boldsymbol{p}) + H_I^{(2)}(K))\Psi\|^2) \leq \|H(K, \boldsymbol{p})\Psi\|^2 + d(\boldsymbol{p})\|\Psi\|^2$$

$$\Psi \in D(H_0^{EM}) \tag{2.22}$$

with c-numbers $c_3(\boldsymbol{p})$ and $d(\boldsymbol{p})$ independent of K. It follows from (2.22) that $H(K, \boldsymbol{p})$ is a closed operator on $D(H_0^{EM})$, which, together with the essential self-adjointness, implies the self-adjointness of $H(K, \boldsymbol{p})$ with $D(H(K, \boldsymbol{p})) = D(H_0^{EM})$. It is clear that $H(K, \boldsymbol{p})$ is bounded below if $m > \delta m(K)$.

Remark. In order to decide whether or not $H(K, \boldsymbol{p})$ with $\delta m(K) > m$ is bounded below, more precise estimates seem to be needed. In the case $\delta m(K) > m$, however, one finds an unphysical solution to the Heisenberg equation (cf Norton and Watson (1959), see also remark on lemma 3.1 of this paper). Physically, therefore, it would be sufficient to consider the case $m > \delta m(K)$.

Lemma 2.1. Let $m > \delta m(K)$ and

$$\hat{H}(K, \boldsymbol{p}) = H(K, \boldsymbol{p}) - E(K, \boldsymbol{p}) \tag{2.23}$$

where $E(K, \boldsymbol{p})$ is the infimum of $\sigma(H(K, \boldsymbol{p}))$, the spectrum of $H(K, \boldsymbol{p})$:

$$E(K, \boldsymbol{p}) = \inf \sigma(H(K, \boldsymbol{p})). \tag{2.24}$$

Then, there exists a c-number $c(\boldsymbol{p})$ independent of K such that

$$\|(H_0^{EM})^{1/2}\Psi\| \leq c(\boldsymbol{p})\|(\hat{H}(K, \boldsymbol{p}) + 1)^{1/2}\Psi\| \qquad \Psi \in D(\hat{H}(K, \boldsymbol{p})^{1/2}) \tag{2.25}$$

and

$$\|\hat{H}(K, \boldsymbol{p})^{1/2}\Psi\| \leq c(\boldsymbol{p})\|(H_0^{EM} + 1)^{1/2}\Psi\| \qquad \Psi \in D((H_0^{EM})^{1/2}). \tag{2.26}$$

In particular, we have

$$D((H_0^{EM})^{1/2}) = D(\hat{H}(K, \boldsymbol{p})^{1/2}). \tag{2.27}$$

Proof. By (2.19) and (2.20) we have

$$\|\hat{H}(K,\boldsymbol{p})\Psi\| \leqslant c(\boldsymbol{p})\|(H_0^{\mathrm{EM}}+1)\Psi\| \tag{2.28}$$

for all $\Psi \in D(H_0^{\mathrm{EM}})$ with $c(\boldsymbol{p})$ independent of K. Inequality (2.25) (respectively (2.26)) follows from (2.22) (respectively (2.28)) and Reed and Simon (1975) (p 168, theorem X18). Equation (2.27) is clear by (2.25) and (2.26).

3. Dressed one-electron states and spectrum of the Hamiltonians

From now on we assume $m > \delta m(0)$. Therefore, we have $m > \delta m(K)$ for all $K \geqslant 0$.

In this section we shall prove the existence and uniqueness (respectively absence) of DES of $H(K,\boldsymbol{p})$ with $K>0$ (respectively $K=0$ and $\boldsymbol{p}\neq 0$) and analyse the spectra of both Hamiltonians.

3.1. *The Heisenberg field*

The Heisenberg field in $\mathscr{H}(\boldsymbol{p})$ is given by

$$\boldsymbol{A}(f,t|K,\boldsymbol{p}) = \exp(\mathrm{i}tH(K,\boldsymbol{p}))\boldsymbol{A}(f)\exp(-\mathrm{i}tH(K,\boldsymbol{p})) \qquad \hat{f}/\sqrt{\omega}, \hat{f}/\omega \in L^2(R^3) \tag{3.1}$$

which, by (2.17), (2.18) and (2.27), is well defined on $D(\hat{H}(K,\boldsymbol{p})^{1/2})$. Formally, $\boldsymbol{A}(f,t|K,\boldsymbol{p})$ satisfies the equation (summation over repeated indices is understood)

$$\left(\frac{\partial^2}{\partial t^2}-\Delta\right)A_\mu(\boldsymbol{x},t|K,\boldsymbol{p}) = -\frac{e}{m-\delta m(K)}\rho_{\mu\nu}(\boldsymbol{x}|K)(p_\nu + eA_\nu(\rho_K,t|K,\boldsymbol{p})) \qquad \mu=1,2,3 \tag{3.2}$$

where

$$\rho_{\mu\nu}(\boldsymbol{x}|K) = (2\pi)^{-3/2}\int \mathrm{d}^3\boldsymbol{k}\, d_{\mu\nu}(\boldsymbol{k})\hat{\rho}_K(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}) \tag{3.3}$$

with

$$d_{\mu\nu}(\boldsymbol{k}) = \delta_{\mu\nu} - k_\mu k_\nu/\boldsymbol{k}^2 \tag{3.4}$$

and $\boldsymbol{A}(\boldsymbol{x},t|K,\boldsymbol{p})$ is the symbolic notation of the field given by

$$\boldsymbol{A}(f,t|K,\boldsymbol{p}) = \int \mathrm{d}^3\boldsymbol{x}\,\boldsymbol{A}(\boldsymbol{x},t|K,\boldsymbol{p})f(\boldsymbol{x}). \tag{3.5}$$

We shall explicitly construct $\boldsymbol{A}(f,t|K,\boldsymbol{p})$ given by (3.1) by solving the equation (3.2). To do this in a rigorous manner we need some lemmas, which go in parallel with Arai (1981b) (lemmas 4.1–4.9). We refer to Arai (1981b) as I.

Let

$$D(z|K) = m - \delta m(K) - \tfrac{2}{3}e^2\int \mathrm{d}^3\boldsymbol{k}\,\frac{\hat{\rho}_K(\boldsymbol{k})^2}{z-\boldsymbol{k}^2} \qquad K \geqslant 0. \tag{3.6}$$

Lemma 3.1. (1) $D(z|K)$ has no zeros in $\mathbb{C}\backslash[K,\infty)$.

(2) $$D_\pm(s|K) \equiv \lim_{\varepsilon\to+0} D(s\pm \mathrm{i}\varepsilon|K) \tag{3.7}$$

exist for each $s \in (K, \infty)$ and are continuous in s. Furthermore,

$$\inf_{s \in (K,\infty)} |D_\pm(s|K)| > c > 0 \tag{3.8}$$

with some constant c independent of K.

The proof is straightforward (cf I (lemma 4.4)).

Remark. If $\delta m(K) > m$, then $D(z|K)$ has a unique simple zero in $(-\infty, 0)$, which is the origin of an unphysical solution.

Let $M_\alpha(R^3)$, $\alpha \in R^3$, be the Hilbert spaces given by

$$M_\alpha(R^3) = \{f \,|\, \|f\|_\alpha \equiv \|\omega^\alpha f\|_0 < \infty\}. \tag{3.9}$$

In I (lemmas 4.1 and 4.2) it was proved that the operator G given by

$$(Gf)(\boldsymbol{k}) = \lim_{\varepsilon \to +0} \int \mathrm{d}^3 k' \frac{f(\boldsymbol{k}')}{(|\boldsymbol{k}||\boldsymbol{k}'|)^{1/2}(\boldsymbol{k}^2 - \boldsymbol{k}'^2 + \mathrm{i}\varepsilon)} \tag{3.10}$$

is a bounded operator on $M_{-1/2}(R^3)$ and $M_0(R^3)$. Let $T_{\mu\nu,K}$, $\mu, \nu = 1, 2, 3$, $K \geqslant 0$, be operators given by

$$T_{\mu\nu,K} f = \delta_{\mu\nu} f + eQ_K \sqrt{\omega} G \sqrt{\omega} d_{\mu\nu} \hat{\rho}_K f \tag{3.11}$$

where

$$Q_K(\boldsymbol{k}) = e\hat{\rho}_K(\boldsymbol{k}) / D_+(\boldsymbol{k}^2|K). \tag{3.12}$$

Then, we have the following.

Lemma 3.2. Each $T_{\mu\nu,K}$ is a bounded operator on $M_\alpha(R^3)$ for $\alpha = -1, \pm\frac{1}{2}, 0$. Furthermore we have the following:

(1)
$$T^*_{\alpha\beta,K} d_{\beta\mu} T_{\mu\nu,K} = d_{\alpha\nu} \tag{3.13}$$

where $T^*_{\alpha\beta,K}$ denotes the adjoint operator of $T_{\alpha\beta,K}$ in $M_0(R^3)$.

(2)
$$e^{(r)}_\alpha T_{\alpha\beta,K} d_{\beta\mu} T^*_{\mu\nu,K} e^{(s)}_\nu = \delta_{rs}. \tag{3.14}$$

(3)
$$T^*_{\mu\nu,K} d_{\nu\alpha} \omega^{-2} Q_K = (e/m) d_{\mu\alpha} \omega^{-2} \hat{\rho}_K. \tag{3.15}$$

(4) If h is a rotation-invariant function on R^3, then we have

$$T^*_{\alpha\beta,K} d_{\beta\mu} h T_{\mu\nu,K} = \overline{T^*_{\alpha\beta,K} d_{\beta\mu} h \bar{T}_{\mu\nu,K}} \tag{3.16}$$

$$T^*_{\alpha\beta,K} d_{\beta\mu} Q_K h = \overline{T^*_{\alpha\beta,K} d_{\beta\mu} \bar{Q}_K h} \qquad \text{AE} \tag{3.17}$$

where the bar denotes the operator defined by

$$\bar{A} f = \overline{A \bar{f}}. \tag{3.18}$$

(5)
$$[\omega^2, T_{\mu\nu,K}] = e(d_{\mu\nu}\hat{\rho}_K, \cdot)_0 Q_K \tag{3.19}$$

$$T_{\mu\nu,K} \hat{\rho}_K = \delta_{\mu\nu} e^{-1}(m - \delta m(K)) Q_K. \tag{3.20}$$

The proof is similar to those in I (lemma 4.9) and is omitted.

Let $\hat{A}_\mu$ and $\hat{\pi}_\mu$ be the Fourier transforms of A_μ and π_μ, respectively:

$$\hat{A}_\mu(f)=A_\mu(\hat{f}) \qquad \hat{\pi}_\mu(f)=\pi_\mu(\hat{f}). \tag{3.21}$$

We define

$$b^{(r)}(f|K,p)=(1/\sqrt{2})[\hat{A}_\mu(T^*_{\mu\alpha,K}e^{(r)}_\alpha\sqrt{\omega}f)+\mathrm{i}\hat{\pi}_\mu(T^*_{\mu\alpha,K}e^{(r)}_\alpha f/\sqrt{\omega})+p_\nu(Q_K e^{(r)}_\nu/\sqrt{\omega},f)_{-1/2}] \tag{3.22}$$

$$b^{(r)*}(f|K,p)=(1/\sqrt{2})[\hat{A}_\mu(\overline{T^*_{\mu\alpha,K}\tilde{e}^{(r)}_\alpha\sqrt{\omega}\bar{f}})-\mathrm{i}\hat{\pi}_\mu(\overline{T^*_{\mu\alpha,K}\tilde{e}^{(r)}_\alpha\bar{f}/\sqrt{\omega}})\ +p_\nu(\bar{Q}_K e^{(r)}_\nu/\sqrt{\omega},f)_{-1/2}]. \tag{3.23}$$

By lemma 3.2, $b^{(r)}(f|K,p)$ and $b^{(r)*}(f|K,p)$, $f\in M_{-1/2}(R^3)\cap M_0(R^3)$, are well defined on $\mathscr{F}_0^{\mathrm{EM}}$, the subspace of finite particle vectors in $\mathscr{F}^{\mathrm{EM}}$, and leave it invariant, satisfying

$$(b^{(r)}(f|K,p)\Psi,\Phi)=(\Psi,b^{(r)*}(\bar{f}|K,p)\Phi) \tag{3.24}$$

$$[b^{(r)}(\bar{f}|K,p),b^{(s)*}(g|K,p)]\Psi=(f,g)_0\delta_{rs}\Psi \tag{3.25}$$

for all Ψ, Φ in $\mathscr{F}_0^{\mathrm{EM}}$ and all f, g in $M_{-1/2}(R^3)\cap M_0(R^3)$; all the other commutators vanish. The commutation relations (3.25) follow from (3.14). Furthermore, we have the following lemma.

Lemma 3.3. (1) Let $b^{(r)\#}(f|K,p)$ denote either $b^{(r)}(f|K,p)$ or $b^{(r)*}(f|K,p)$. Then

$$\|b^{(r)\#}(f|K,p)\Psi\|\le c(p)(\|f\|_{-1/2}+\|f\|_0)\|(\hat{H}(K,p)+1)^{1/2}\Psi\| \tag{3.26}$$

for all Ψ in $D(\hat{H}(K,p)^{1/2})$ and f in $M_{-1/2}(R^3)\cap M_0(R^3)$ with some c-number $c(p)$ independent of K.

(2) Let f be in $M_{-1/2}(R^3)\cap M_0(R^3)\cap M_1(R^3)$. Then, $b^{(r)\#}(f|K,p)$ maps $D(\hat{H}(K,p)^{3/2})$ into $D(\hat{H}(K,p))$ and

$$[\hat{H}(K,p),b^{(r)\#}(f|K,p)]\Psi=\pm b^{(r)\#}(\omega f|K,p)\Psi \qquad \Psi\in D(\hat{H}(K,p)^{3/2}) \tag{3.27}$$

where + (respectively −) corresponds to $b^{(r)*}(\cdot|K,p)$ (respectively $b^{(r)}(\cdot|K,p)$).

Proof. Inequality (3.26) follows from estimates (2.17), (2.18), lemma 3.2 (boundedness of $T_{\mu\nu,K}$ on $M_\alpha(R^3)$, $\alpha=-1,\pm\frac{1}{2},0$) and (2.25). By the canonical commutation relations for $a^{(r)\#}(\cdot)$ we first prove (3.27) for Ψ in $\mathscr{F}_0^{\mathrm{EM}}\cap D(\hat{H}(K,p)^{3/2})$ and then use a limiting argument to extend the result to all Ψ in $D(\hat{H}(K,p)^{3/2})$.

Theorem 3.1. The Heisenberg field defined by (3.1) has the explicit form

$$\begin{aligned}A_\mu(f,t|K,p)=&\frac{1}{\sqrt{2}}\sum_{r=1}^{2}\left[b^{(r)*}\left(\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\sqrt{\omega}}e^{(r)}_\nu\bar{T}_{\mu\nu,K}\hat{f}\middle|K,p\right)+b^{(r)}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{\omega}}e^{(r)}_\nu T_{\mu\nu,K}\bar{\hat{f}}\middle|K,p\right)\right]\\&-(e/m)p_\nu(d_{\mu\nu}\hat{\rho}_K,\hat{f})_{-1}\\&\mu=1,2,3 \qquad \hat{f}\in M_{-1}(R^3)\cap M_{-1/2}(R^3).\end{aligned} \tag{3.28}$$

Proof. Since we have established lemmas 3.2 and 3.3, the proof can be done in the same way as in I (theorem 4.1): the initial condition is checked by (3.13) and (3.15)–(3.17), and the Heisenberg equation follows from lemma 3.3.

Remark. By (3.19) and (3.20) we can show that the Heisenberg field $A(f, t|K, p)$ with $f \in \mathscr{S}(R^3)$ satisfies the equation (3.2) on $\mathscr{F}_0^{\mathrm{EM}}$ in the sharp-time operator-valued distribution sense with the time derivative being taken in the strong topology.

3.2. *Asymptotic fields*

Let

$$a^{(r)\#}(f, t)|K, p) = \exp(\mathrm{i}(tH(K, p))\exp(-\mathrm{i}tH_0^{\mathrm{EM}})a^{(r)\#}(f)\exp(\mathrm{i}tH_0^{\mathrm{EM}})\exp(-\mathrm{i}tH(K, p)) \qquad r = 1, 2 \qquad K \geq 0 \tag{3.29}$$

with f in $M_{-1/2}(R^3) \cap M_0(R^3)$. By (2.27), $a^{(r)\#}(f, t|K, p)$ is well defined on $D(\hat{H}(K, p)^{1/2})$.

Theorem 3.2. Let Ψ be in $D(\hat{H}(K, p)^{1/2})$. Then, the strong limits

$$s - \lim_{t \to \pm\infty} a^{(r)\#}(f, t|K, p)\Psi \equiv a^{(r)\#}_{\mathrm{out/in}}(f|K, p)\Psi \qquad K \geq 0 \tag{3.30}$$

exist and are given explicitly by

$$a^{(r)\#}_{\mathrm{in}}(f|K, p) = b^{(r)\#}(f|K, p) \tag{3.31}$$

$$a^{(r)}_{\mathrm{out}}(f|K, p) = \sum_{s=1}^{2} b^{(s)}(L_K^{(s,r)}f|K, p) \tag{3.32}$$

$$a^{(r)*}_{\mathrm{out}}(f|K, p) = \sum_{s=1}^{2} b^{(s)*}(\bar{L}_K^{(s,r)}f|K, p) \tag{3.33}$$

where

$$L_K^{(s,r)}f = \delta_{sr}f - \mathrm{i}\pi e\omega\hat{\rho}_K Q_K e_\mu^{(s)}[fe_\mu^{(r)}] \tag{3.34}$$

and the function $[f]$ is defined by

$$[f](k) = \int_{S^2} \mathrm{d}\Omega\, f(|k|\Omega). \tag{3.35}$$

The proof is similar to that of I (theorem 5.1) and is omitted.

Remark. If $\operatorname{supp} f \subset \{|k| < K\}$ $(K > 0)$, then

$$a^{(r)\#}(f, t|K, p) = a^{(r)\#}(f).$$

Thus, in the proof of theorem 3.2, we need only consider the case when $\operatorname{supp} f \subset \{|k| \geq K\}$. Note also that, if $\operatorname{supp} f \subset \{|k| < K\}$ $(K > 0)$, then we have from (3.22) and (3.23)

$$b^{(r)\#}(f|K, p) = a^{(r)\#}(f).$$

3.3. DES, *infrared cut-off and spectrum of the Hamiltonians*

Since we have established theorems 3.1 and 3.2, we can prove the following (see I (lemmas 6.2–6.4)).

Lemma 3.4. (1) Either $\sigma_p(H(K,\boldsymbol{p}))=\varnothing$ or $\sigma_p(H(K,\boldsymbol{p}))=\{E(K,\boldsymbol{p})\}$ holds, where $\sigma_p(H(K,\boldsymbol{p}))$ denotes the point spectrum of $H(K,\boldsymbol{p})$.

(2) The ground state of $H(K,\boldsymbol{p})$, i.e., the DES with electron momentum $\boldsymbol{p}$, if it exists, is unique up to scalar multiples.

(3) Let $K\geqslant 0$ and $\boldsymbol{p}$ be fixed. Then, Ψ is the DES of $H(K,\boldsymbol{p})$ if and only if $b^{(r)}(f|K,\boldsymbol{p})\Psi=0$, $r=1,2$, for all f in $M_{-1/2}(R^3)\cap M_0(R^3)$.

Keeping the above lemma in mind, we shall prove that the infrared cut-off (respectively no infrared cut-off) implies the existence (respectively absence) of DES. We first have the following.

Theorem 3.3 (absence of DES). Let $\boldsymbol{p}\neq 0$. Then, $H(0,\boldsymbol{p})$ has no DES.

Proof. Suppose there exists a non-zero vector Ψ in $\mathscr{H}(\boldsymbol{p})$ such that

$$\hat{H}(0,\boldsymbol{p})\Psi=0.$$

Then, by lemma 3.4(3), we have

$$b^{(r)}(f|0,\boldsymbol{p})\Psi=0 \qquad r=1,2 \qquad f\in M_{-1/2}(R^3)\cap M_0(R^3). \tag{3.36}$$

Let

$$F^{(r)}(f)=p_\mu(Q_0e_\mu^{(r)}/\sqrt{\omega},f)_{-1/2}. \tag{3.37}$$

Then, by lemma 3.2, (3.22) and (3.36), we can show that $F^{(r)}(f)$ defines a continuous linear functional on $M_0(R^3)$. Therefore, by the Riesz lemma, there exists a non-zero vector $\xi^{(r)}\in M_0(R^3)$ such that

$$F^{(r)}(f)=(\xi^{(r)},f)_0 \qquad f\in M_0(R^3).$$

Comparing with (3.37), we get

$$\xi^{(r)}=\omega^{-3/2}Q_0e_\mu^{(r)}p_\mu.$$

This, however, is a contradiction because the RHS is not contained in $M_0(R^3)$. Thus, $H(0,\boldsymbol{p})(\boldsymbol{p}\neq 0)$ has no DES.

We next proceed to construct the DES of $H(K,\boldsymbol{p})$ with $K>0$ or of $H(0,\boldsymbol{0})$. Let

$$W_+^{(r,s)}(K)f=\tfrac{1}{2}\left(\frac{1}{\sqrt{\omega}}e_\mu^{(r)}T^*_{\mu\nu,K}e_\nu^{(s)}\sqrt{\bar{\omega}}+\sqrt{\omega}e_\mu^{(r)}T^*_{\mu\nu,K}e_\nu^{(s)}\frac{1}{\sqrt{\omega}}\right)f \tag{3.38}$$

$$W_-^{(r,s)}(K)f=\tfrac{1}{2}\left(\frac{1}{\sqrt{\omega}}e_\mu^{(r)}T^*_{\mu\nu,K}\tilde{e}_\nu^{(s)}\sqrt{\omega}-\sqrt{\omega}e_\mu^{(r)}T^*_{\mu\nu,K}\tilde{e}_\nu^{(s)}\frac{1}{\sqrt{\omega}}\right)\tilde{f}. \tag{3.39}$$

Then, by lemma 3.2, $W_\pm^{(r,s)}(K)$ are bounded operators on $M_\alpha(R^3)$ for $\alpha=-\frac{1}{2},0$. We can rewrite (3.22) and (3.23) as

$$b^{(r)}(f|K,\boldsymbol{p})=\sum_{s=1}^{2}[a^{(s)*}(W_-^{(s,r)}(K)f)+a^{(s)}(W_+^{(s,r)}(K)f)]+\frac{1}{\sqrt{2}}p_\mu(Q_Ke_\mu^{(r)}/\sqrt{\omega},f)_{-1/2} \tag{3.40}$$

$$b^{(r)*}(f|K,\boldsymbol{p})=\sum_{s=1}^{2}[a^{(s)*}(\bar{W}_-^{(s,r)}(K)f)+a^{(s)}(\bar{W}_-^{(s,r)}(K)f)]+\frac{1}{\sqrt{2}}p_\mu(\bar{Q}_Ke_\mu^{(r)}/\sqrt{\omega},f)_{-1/2}. \tag{3.41}$$

The properties of $T_{\mu\nu,K}$ given in lemma 3.2 read

$$W_+(K)^* W_+(K) - W_-(K)^* W_-(K) = I \tag{3.42}$$

$$\overline{W_+(K)^*} W_-(K) - \overline{W_-(K)^*} W_+(K) = 0 \tag{3.43}$$

$$W_+(K) W_+(K)^* - \overline{W_-(K)}\, \overline{W_-(K)^*} = I \tag{3.44}$$

$$W_-(K) W_+(K)^* - \overline{W_+(K)}\, \overline{W_-(K)^*} = 0 \tag{3.45}$$

where $W_\pm(K)$ are bounded operators on the Hilbert spaces

$$N_\alpha(R^3) = (M_\alpha(R^3))^2 \tag{3.46}$$

for $\alpha = -\frac{1}{2}$, 0, defined by

$$W_\pm(K) = (W_\pm^{(r,s)}(K)). \tag{3.47}$$

Lemma 3.5. For all $K \geqslant 0$, $W^{(r,s)}(K)$ is a Hilbert–Schmidt operator on $M_0(R^3)$.

Proof. By direct computations, we have

$$(W_-^{(r,s)}(K)f)(\boldsymbol{k}) = \int \mathrm{d}^3\boldsymbol{k}'\, w_K(\boldsymbol{k}, \boldsymbol{k}')f(\boldsymbol{k}')$$

where

$$w_K(\boldsymbol{k}, \boldsymbol{k}') = \frac{e\hat{\rho}_K(\boldsymbol{k})\bar{Q}_K(\boldsymbol{k}')e_\mu^{(r)}(\boldsymbol{k})e_\mu^{(s)}(\boldsymbol{k}')}{2(|\boldsymbol{k}| + |\boldsymbol{k}'|)(|\boldsymbol{k}||\boldsymbol{k}'|)^{1/2}}.$$

It can be easily seen that $w_K(\cdot, \cdot) \in L^2(R^6)$ for all $K \geqslant 0$. Thus the lemma follows.

We can see from (3.42) and (3.44) that, for all $K \geqslant 0$, $W_+(K)^{-1}$ exists as a bounded operator on $N_0(R^3)$. Let

$$C(K) = W_-(K) W_+(K)^{-1}. \tag{3.48}$$

Then, by lemma 3.5, each $C^{(r,s)}(K)$ is a Hilbert–Schmidt operator on $M_0(R^3)$.

Lemma 3.6. Let $C_K^{(r,s)}(\boldsymbol{k}, \boldsymbol{k}')$ be the Hilbert–Schmidt kernel of $C^{(r,s)}(K)$. Then, we have

$$C_K^{(r,s)}(\boldsymbol{k}, \boldsymbol{k}') = C_K^{(s,r)}(\boldsymbol{k}', \boldsymbol{k}). \tag{3.49}$$

Proof. Equation (3.49) is equivalent to $C(K)^* = \overline{C(K)}$, i.e.,

$$(W_+(K)^{-1})^* W_-(K)^* = \overline{W_-(K)}\, \overline{W_+(K)^{-1}}$$

which is in turn equivalent to (3.43).

Let $K > 0$ and

$$V(K, \boldsymbol{p}) = -\tfrac{1}{2} \sum_{r,s=1}^{2} \int \mathrm{d}^3\boldsymbol{k}\, \mathrm{d}^3\boldsymbol{k}'\, a^{(r)*}(\boldsymbol{k}) C_K^{(r,s)}(\boldsymbol{k}, \boldsymbol{k}') a^{(s)*}(\boldsymbol{k}')$$
$$- \sum_{r=1}^{2} a^{(r)*}(\overline{((W_+(K)^{-1*} u_{\mu,K})^{(r)}}) p_\mu \tag{3.50}$$

where $u_{\mu,K}$ is a function in $N_0(R^3)$ given by

$$u^{(r)}_{\mu,K} = \omega^{-3/2} Q_K e^{(r)}_{\mu}/\sqrt{2}. \tag{3.51}$$

We can see that the vector

$$\Psi_1(K, \boldsymbol{p}) = c(K, \boldsymbol{p}) \sum_{n=0}^{\infty} \frac{V(K, \boldsymbol{p})^n}{n!} \Omega_0 \tag{3.52}$$

is well defined, where Ω_0 is the Fock vacuum in $\mathscr{F}^{\mathrm{EM}}$ and $c(K, \boldsymbol{p})$ is a c-number such that $\|\Psi_1(K, \boldsymbol{p})\| = 1$.

Theorem 3.4. For each $\boldsymbol{p} \in R^3$, the DES of $H(K, \boldsymbol{p})$ with $K > 0$ exists and is given by (3.52) up to scalar multiples.

Proof. By lemma 3.2(2)–(3), we need only to show that

$$b^{(r)}(f|K, \boldsymbol{p})\Psi_1(k, \boldsymbol{p}) = 0 \qquad r = 1, 2 \qquad f \in M_{-1/2}(R^3) \cap M_0(R^3). \tag{3.53}$$

We can see that

$$\left[\sum_{s=1}^{2} a^{(s)}(W^{(s,r)}_{+}(K)f),\, V(K, \boldsymbol{p})\right]$$
$$= -\left\{\sum_{s=1}^{2} a^{(s)*}(W^{(s,r)}_{-}(K)f) + (u^{(r)}_{\mu,K}, f)_0 p_\mu\right\} \qquad \text{on } \mathscr{F}^{\mathrm{EM}}_0,$$

so that

$$\sum_{s=1}^{2} a^{(s)}(W^{(s,r)}_{+}(K)f)V(K, \boldsymbol{p})^n \Omega_0$$
$$= -n\left\{\sum_{s=1}^{2} a^{(s)*}(W^{(s,r)}_{-}(K)f) + (u^{(r)}_{\mu,K}, f)_0 p_\mu\right\} V(K, \boldsymbol{p})^{n-1}\Omega_0.$$

Thus, we get (3.53).

Corollary 3.1. The DES of $H(0, \mathbf{0})$ exists and is unique up to scalar multiples.

Proof. We note that, for $\boldsymbol{p} = 0$, $V(K, \boldsymbol{p})$ given by (3.50) is well defined even for $K = 0$ and so is $\Psi_1(0, \mathbf{0})$ by (3.52). In the same way as above we can show that $\Psi_1(0, \mathbf{0})$ is the DES of $H(0, \mathbf{0})$.

Corollary 3.2. (1) Let $K > 0$. Then, we have

$$\sigma(H(K, \boldsymbol{p})) = [E(K, \boldsymbol{p}), \infty) \qquad \sigma_{\mathrm{p}}(H(K, \boldsymbol{p})) = \{E(K, \boldsymbol{p})\}. \tag{3.54}$$

The eigenvalue $E(K, \boldsymbol{p})$ is simple and is given by

$$E(K, \boldsymbol{p}) = \frac{1}{2(m - \delta m(K))}\{(\boldsymbol{p} - \boldsymbol{R}(K, \boldsymbol{p}))^2 - \tfrac{1}{2}e^2(v_{\mu,K}, C(K)v_{\mu,K})_{N_0(R^3)}\} \tag{3.55}$$

where

$$v^{(r)}_{\mu,K} = \hat{\rho}_K e^{(r)}_{\mu}/\sqrt{2\omega} \qquad R_\mu(K, \boldsymbol{p}) = p_\nu(u_{\nu,K}, W_+(K)^{-1}v_{\mu,K})_{N_0(R^3)}. \tag{3.56}$$

(2) $$\sigma(H(0, \mathbf{0})) = [E_0, \infty) \qquad \sigma_{\mathrm{p}}(H(0, \mathbf{0})) = \{E_0\}. \tag{3.57}$$

The eigenvalue E_0 is simple and is given by

$$E_0 = -\frac{e^2}{4(m-\delta m(0))}(v_{\mu,0}, C(0)v_{\mu,0})_{N_0(R^3)}. \tag{3.58}$$

Proof. (3.54) (respectively (3.57)) follows from theorem 3.4 (respectively corollary 3.1), lemma 3.4(1) and Arai (1981a) (proposition 4.1). The eigenvalue $E(K, \boldsymbol{p})$ can be computed from the identity

$$E(K, \boldsymbol{p}) = (\Omega_0, H(K, \boldsymbol{p})\Psi_1(K, \boldsymbol{p}))/(\Omega_0, \Psi_1(K, \boldsymbol{p})).$$

Remark. It can be seen that $\hat{H}(K, \boldsymbol{p})$ with $K > 0$ and $\hat{H}(0, \boldsymbol{0})$ are unitarily equivalent to H_0^{EM}.

Corollary 3.3. The spectrum of $H(0, \boldsymbol{p})$ with $\boldsymbol{p} \neq 0$ is purely continuous:

$$\sigma(H(0, \boldsymbol{p})) = [E(0, \boldsymbol{p}), \infty) \qquad \sigma_{\mathrm{p}}(H(0, \boldsymbol{p})) = \varnothing. \tag{3.59}$$

Proof. By estimates similar to (2.19) and (2.20), one can easily see that

$$\lim_{K \to 0} \|(H(K, \boldsymbol{p}) - z)^{-1}\| = \|(H(0, \boldsymbol{p}) - z)^{-1}\| \tag{3.60}$$

for all $z \in \rho(H(K, \boldsymbol{p})) \cap \rho(H(0, \boldsymbol{p}))$, where $\rho(H(K, \boldsymbol{p}))$ denotes the resolvent set of $H(K, \boldsymbol{p})$. Since $(E(K, \boldsymbol{p}) + c)^{-1} = \|(H(K, \boldsymbol{p}) + c)^{-1}\|$ for all $K \geqslant 0$ and sufficiently large $c > 0$, we get

$$\lim_{K \to 0} E(K, \boldsymbol{p}) = E(0, \boldsymbol{p}). \tag{3.61}$$

Using (3.61), we can prove in the same way as in the proof of (3.60) that

$$\lim_{K \to 0} \|(\hat{H}(K, \boldsymbol{p}) - z)^{-1}\| = \|(\hat{H}(0, \boldsymbol{p}) - z)^{-1}\|$$

for all $z \in \mathbb{C} \backslash [0, \infty)$. Since $\sigma(\hat{H}(K, \boldsymbol{p})) = [0, \infty)$ by corollary 3.2, it follows from a standard theorem (see Reed and Simon (1972) (theorem VIII.23)) that $[0, \infty) \subset \sigma(\hat{H}(0, \boldsymbol{p}))$. On the other hand, $\hat{H}(0, \boldsymbol{p})$ is positive. Thus we get the first part of (3.59). The second part of (3.59) follows from lemma 3.4(1) and theorem 3.3.

4. Removal of infrared cut-off

We can now construct the Wightman distributions and remove the infrared cut-off.

Lemma 4.1. There exists a c-number $c(\boldsymbol{p})$ independent of K such that

$$\left.\begin{aligned} |(\Psi, A_\mu(f)\Psi)| &\leqslant c(\boldsymbol{p}) \max\{\|\hat{f}\|_{-1}, \|\hat{f}\|_{-1/2}\}(\Psi, (\hat{H}(K, \boldsymbol{p}) + 1)\Psi) \\ |(\Psi, \pi_\mu(f)\Psi)| &\leqslant c(\boldsymbol{p}) \max\{\|\hat{f}\|_0, \|\hat{f}\|_{1/2}\}(\Psi, (\hat{H}(K, \boldsymbol{p}) + 1)\Psi) \end{aligned}\right\} \qquad \Psi \in D(\hat{H}(K, \boldsymbol{p})).$$

Proof. Use (2.17), (2.18) and (2.25).

Lemma 4.2. Let $f \in \mathcal{S}(R^3)$. Then, $A_\mu(f)$, $\mu = 1, 2, 3$, leave $C^\infty(H(K, \boldsymbol{p}))$ invariant and for any $\alpha > 0$ there exist a constant β and a $\mathcal{S}(R^3)$-norm $|||f|||$ independent of K

such that

$$\|(\hat{H}(K,\boldsymbol{p})+1)^{\alpha/2}A_\mu(f)\Psi\| \leqslant |||f|||\,\|(\hat{H}(K,\boldsymbol{p})+1)^{\beta/2}\Psi\|$$

for all Ψ in $C^\infty(H(K,\boldsymbol{p}))$.

Proof. By direct computation we can show that

$$[\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]^{2n}A_\mu(f)$$
$$= A_{\nu_n}(J^K_{\nu_n\nu_{n-1}}J^K_{\nu_{n-1}\nu_{n-2}}\dots J^K_{\nu_1\mu}f) + p_{\nu_n}\alpha^K_{\nu_n\nu_{n-1}}(J^K_{\nu_{n-1}\nu_{n-2}}J^K_{\nu_{n-2}\nu_{n-3}}\dots J^K_{\nu_1\mu}f)$$
$$[\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]^{2n-1}A_\mu(f) = -\mathrm{i}\pi_{\nu_{n-1}}(J^K_{\nu_{n-1}\nu_{n-2}}\dots J^K_{\nu_1\mu}f) \qquad n \geqslant 1$$

where $[\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]^nA_\mu(f)$ is defined recursively by

$$[\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]A_\mu(f) = [\hat{H}(K,\boldsymbol{p}), A_\mu(f)]$$
$$[\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]^nA_\mu(f) = [\hat{H}(K,\boldsymbol{p}), [\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]^{n-1}A_\mu(f)] \qquad n \geqslant 2$$

and

$$J^K_{\mu\nu}f = \delta_{\mu\nu}(-\Delta f) + e\alpha^K_{\mu\nu}(f)\rho_K$$

$$\alpha^K_{\mu\nu}(f) = \frac{e}{m-\delta m(K)}(d_{\mu\nu}\hat{\rho}_K, \hat{f})_0.$$

One can see that, for $\alpha \geqslant -1$,

$$\|(J^K_{\nu_n\nu_{n-1}}\dots J^K_{\nu_1\mu}f)\hat{}\|_\alpha \leqslant \sum_{j=0}^{2n+\alpha} C^{(n)}_{j,\alpha}\|\hat{f}\|_j$$

with constants $C^{(n)}_{j,\alpha}$ independent of K. Therefore, by lemma 4.1, we get

$$|(\Psi, [\operatorname{Ad}\hat{H}(K,\boldsymbol{p})]^nA_\mu(f)\Psi)| < |||f|||_n(\Psi, (\hat{H}(K,\boldsymbol{p})+1)\Psi)$$
$$n \geqslant 1 \qquad \Psi \in D(\hat{H}(K,\boldsymbol{p}))$$

with some $\mathscr{S}(\boldsymbol{R}^3)$-norm $|||f|||_n$ independent of K. The lemma now follows from the Nelson lemma (see Nelson 1972).

Let $K>0$. By theorem 3.4 and lemma 4.2 we can define the Wightman distributions

$$W^{(n)}_{\mu_1\dots\mu_n}(f_1, t_1, \dots, f_n, t_n|K,\boldsymbol{p})$$
$$= (\Psi_1(K,\boldsymbol{p}), A_{\mu_1}(f_1, t_1|K,\boldsymbol{p})\dots A_{\mu_n}(f_n, t_n|K,\boldsymbol{p})\Psi_1(K,\boldsymbol{p}))$$
$$f_j \in \mathscr{S}(\boldsymbol{R}^3) \qquad j=1,\dots,n \qquad n \geqslant 0. \tag{4.1}$$

Putting

$$\tilde{A}_\mu(f,t|K,\boldsymbol{p}) = \frac{1}{\sqrt{2}}\sum_{r=1}^{2}\left\{b^{(r)*}\left(\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\sqrt{\omega}}e^{(r)}_\mu\bar{T}_{\mu\nu,K}\hat{f}\middle|K,\boldsymbol{p}\right) + b^{(r)}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{\omega}}e^{(r)}_\nu T_{\mu\nu,K}\bar{\hat{f}}\middle|K,\boldsymbol{p}\right)\right\} \tag{4.2}$$

$$C_\mu(f|K,\boldsymbol{p}) = -\frac{e}{m}(d_{\mu\nu}\hat{\rho}_K, \hat{f})_{-1}p_\nu \tag{4.3}$$

we can write

$$A_\mu(f,t|K,\boldsymbol{p}) = \tilde{A}_\mu(f,t|K,\boldsymbol{p}) + C_\mu(f|K,\boldsymbol{p}). \tag{4.4}$$

Therefore we have

$$W^{(n)}_{\mu_1\ldots\mu_n}(f_1, t_1, \ldots, f_n, t_n|K, p)$$

$$= \sum_{2^n \text{ terms}} (\Psi_1(K, p), B_{\mu_1}(f_1, t_1|K, p) \ldots B_{\mu_n}(f_n, t_n|K, p)\Psi_1(K, p)) \qquad (4.5)$$

where $B_\mu(f, t|K, p)$ denotes either $\tilde{A}_\mu(f, t|K, p)$ or $C_\mu(f|K, p)$. Thus, we need only consider

$$\tilde{W}^{(n)}_{\mu_1\ldots\mu_n}(f_1, t_1, \ldots, f_n, t_n|K, p)$$

$$= (\Psi_1(K, p), \tilde{A}_{\mu_1}(f_1, t_1|K, p) \ldots \tilde{A}_{\mu_n}(f_n, t_n|K, p)\Psi_1(K, p)) \qquad (4.6)$$

and $C_\mu(f|K, p)$. It is easy to see that

$$\tilde{W}^{(2n)}_{\mu_1\ldots\mu_{2n}}(f_1, t_1, \ldots, f_{2n}, t_{2n}|K, p)$$

$$= \sum_{\text{comb}} \tilde{W}^{(2)}_{\mu_{i_1}\mu_{j_1}}(f_{i_1}, t_{i_1}, f_{j_1}, t_{j_1}|K, p) \ldots \tilde{W}^{(2)}_{\mu_{i_n}\mu_{j_n}}(f_{i_n}, t_{i_n}, f_{j_n}, t_{j_n}|K, p) \qquad (4.7)$$

$$\tilde{W}^{(2n-1)}_{\mu_n\ldots\mu_{2n-1}}(f_1, t_1, \ldots, f_{2n-1}, t_{2n-1}|K, p) = 0 \qquad n \geq 1 \qquad (4.8)$$

where Σ_{comb} indicates the sum over all $(2n)!/2^n n!$ ways of writing $1, \ldots, 2n$ as n distinct unordered pairs $(i_1, j_1), \ldots, (i_n, j_n)$. The two-point function $\tilde{W}^{(2)}_{\mu\nu}(f, t, g, s|K, p)$ is given by

$$\tilde{W}^{(2)}_{\mu\nu}(f, t, g, s|K, p) = \tfrac{1}{2} \sum_{r=1}^{2} (e^{i\omega t} e^{(r)}_\alpha \bar{T}_{\mu\alpha,K}\hat{f}, e^{i\omega s} e^{(r)}_\beta \bar{T}_{\nu\beta,K}\hat{g})_{-1/2}. \qquad (4.9)$$

Lemma 4.3. (1) Let

$$T_{\mu\nu} \equiv T_{\mu\nu,0}. \qquad (4.10)$$

Then,

$$\lim_{K\to 0} \|(T_{\mu\nu,K} - T_{\mu\nu})f\|_\alpha = 0 \qquad f \in M_\alpha(R^3) \qquad \alpha = -1, \pm\tfrac{1}{2}, 0. \qquad (4.11)$$

(2)

$$\lim_{K\to 0} C_\mu(f|K, p) = C_\mu(f|0, p) \equiv C_\mu(f|p). \qquad (4.12)$$

Proof. We can write

$$(T_{\mu\nu,K} - T_{\mu\nu})f = e(Q_K - Q)\sqrt{\omega}G\sqrt{\omega}d_{\mu\nu}\hat{\rho}f + eQ_K\sqrt{\omega}G\sqrt{\omega}d_{\mu\nu}(\hat{\rho}_K - \hat{\rho})f.$$

By (3.8) we have

$$\sup_k |Q_K(k)| < c$$

for some constant c independent of K. Since $\hat{\rho}_K(k) \to \hat{\rho}(k)$ as $K \to 0$ and

$$\lim_{K\to 0} D_+(k^2|K) = D_+(k^2|0) \equiv D_+(k^2), \qquad (4.13)$$

the first part follows from the dominated convergence theorem. The second part is clear.

Let

$$F = (2/3m)(\sqrt{\omega}G\sqrt{\omega}\hat{\rho}^2 + \|\hat{\rho}/\omega\|_0^2). \qquad (4.14)$$

Then, we have

$$D_+(\boldsymbol{k}^2) = m(1 - e^2 F(\boldsymbol{k})). \tag{4.15}$$

Since F is continuous on R^3 (see lemma 3.1) and $F(\boldsymbol{k}) \to (2/3m)\|\hat{\rho}/\omega\|_0^2$ as $|\boldsymbol{k}| \to \infty$, we get

$$0 < M^2 \equiv \sup_{\boldsymbol{k}} |F(\boldsymbol{k})| < \infty. \tag{4.16}$$

Lemma 4.4. The operator $T_{\mu\nu}$ is analytic with respect to e on the region

$$\Pi_M \equiv \{e \in \mathbb{C} \mid |e| < M^{-1}\} \tag{4.17}$$

where the analyticity is taken in the operator-norm sense on $M_\alpha(R^3)$ for $\alpha = -1, \pm\frac{1}{2}, 0$. The Taylor expansion is given by

$$T_{\mu\nu} = \sum_{n=0}^{\infty} e^{2n} T_{\mu\nu}^{(n)} \tag{4.18}$$

where

$$T_{\mu\nu}^{(0)} = \delta_{\mu\nu} \qquad T_{\mu\nu}^{(n)} f = m^{-1} F^{n-1} \hat{\rho}\sqrt{\omega} G \sqrt{\omega} d_{\mu\nu} \hat{\rho} f \qquad n \geqslant 1. \tag{4.19}$$

Proof. We have from (4.15) and (4.16)

$$\frac{1}{D(\boldsymbol{k}^2)} = \frac{1}{m} \sum_{n=0}^{\infty} e^{2n} F(\boldsymbol{k})^n \qquad e \in \Pi_M \tag{4.20}$$

where the convergence is absolute and uniform. Let

$$U_{\mu\nu}^{(N)} = \sum_{n=0}^{N} e^{2n} T_{\mu\nu}^{(n)}.$$

It is easy to see that

$$(T_{\mu\nu} - U_{\mu\nu}^{(N)}) f = (e^2 F)^N e Q \hat{\rho} \sqrt{\omega} G \sqrt{\omega} d_{\mu\nu} \hat{\rho} f.$$

Noting the boundedness of G on $M_\alpha(R^3)$, $\alpha = -\frac{1}{2}, 0$, we have

$$\sup_f \|(T_{\mu\nu} - U_{\mu\nu}^{(N)}) f\|_\alpha / \|f\|_\alpha \leqslant \text{constant} \times |e| (|e| M)^{2N} \to 0 \ (N \to \infty) \qquad \alpha = -1, \pm\tfrac{1}{2}, 0$$

which imply the lemma.

Theorem 4.1. For all $n \geqslant 0$,

$$\lim_{K \to 0} W_{\mu_1 \cdots \mu_n}^{(n)}(f_1, t_1, \ldots, f_n, t_n | K, \boldsymbol{p}) \equiv W_{\mu_1 \cdots \mu_n}^{(n)}(f_1, t_1, \ldots, f_n, t_n | \boldsymbol{p}) \tag{4.21}$$

exists and is analytic with respect to e on the region Π_M.

Proof. It follows from (4.9) and (4.11) that

$$\begin{aligned} \lim_{K \to 0} \tilde{W}_{\mu\nu}^{(2)}(f, t, g, s | K, \boldsymbol{p}) &= \tfrac{1}{2} \sum_{r=1}^{2} (e^{i\omega t} e_\alpha^{(r)} \bar{T}_{\mu\alpha} \hat{\bar{f}}, e^{i\omega s} e_\beta^{(r)} \bar{T}_{\nu\beta} \hat{g})_{-1/2} \\ &\equiv \tilde{W}_{\mu\nu}^{(2)}(f, t, g, s) \end{aligned} \tag{4.22}$$

which, together with (4.5), (4.7) and (4.12), implies the existence of the limit $K \to 0$

of the Wightman distributions. This proves the first half of the theorem. We have by lemma 4.4

$$\tilde{W}^{(2)}_{\mu\nu}(f, t, g, s) = \sum_{n=0}^{\infty} e^{2n} C^{(n)}_{\mu\nu}(f, t, g, s) \qquad 0 \leq e < M^{-1} \tag{4.23}$$

where

$$C^{(n)}_{\mu\nu}(f, t, g, s) = \tfrac{1}{2} \sum_{j=0}^{n} \sum_{r=1}^{2} (\mathrm{e}^{\mathrm{i}\omega t} e^{(r)}_{\alpha} \bar{T}^{(j)}_{\mu\alpha} \hat{\bar{f}}, \mathrm{e}^{\mathrm{i}\omega s} e^{(r)}_{\beta} \bar{T}^{(n-j)}_{\nu\beta} \hat{g})_{-1/2}. \tag{4.24}$$

Therefore, $\tilde{W}^{(2)}_{\mu\nu}(f, t, g, s)$ has an analytic continuation with respect to e onto the region Π_M. Since $W^{(n)}_{\mu_1 \ldots \mu_n}(f_1, t_1, \ldots, f_n, t_n|\boldsymbol{p})$ can be written as a sum of products of $\tilde{W}^{(2)}_{\mu_i \mu_j}(f_i, t_i, f_j, t_j)$ and $C_{\mu_i}(f_i|\boldsymbol{p})$ (see (4.5)–(4.8)), the second half of the theorem follows.

The Wightman reconstruction theorem permits us to construct a field theory from the sequence $\{W^{(n)}_{\mu_1 \ldots \mu_n}(f_1, t_1, \ldots, f_n, t_n|\boldsymbol{p})\}$, which should be the theory of our model with the infrared cut-off removed. We shall construct it explicitly.

Definitions

(1) The Hilbert space of state vectors:

$$\mathscr{H}^{\mathrm{ren}}(\boldsymbol{p}) = \mathscr{F}^{\mathrm{EM}} \qquad \boldsymbol{p} \in \boldsymbol{R}^3. \tag{4.25}$$

(2) The Heisenberg field:

$$A_\mu(f, t|\boldsymbol{p}) = \frac{1}{\sqrt{2}} \sum_{r=1}^{2} \left\{ a^{(r)*}\left(\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\sqrt{\omega}} e^{(r)}_{\nu} \bar{T}_{\mu\nu} \hat{f}\right) + a^{(r)}\left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{\omega}} e^{(r)}_{\nu} T_{\mu\nu} \hat{\bar{f}}\right) \right\} + C_\mu(f|\boldsymbol{p}) \qquad f \in \mathscr{S}(\boldsymbol{R}^3). \tag{4.26}$$

(3) The Hamiltonian:

$$H(\boldsymbol{p}) = H_0^{\mathrm{EM}} \qquad \boldsymbol{p} \in \boldsymbol{R}^3. \tag{4.27}$$

(4) The DES:

$$\Psi_1(\boldsymbol{p}) = \Omega_0 \qquad \boldsymbol{p} \in \boldsymbol{R}^3. \tag{4.28}$$

Theorem 4.2. (1) The Heisenberg field $\boldsymbol{A}(f, t|\boldsymbol{p})$ satisfies the equation

$$\left(\frac{\partial^2}{\partial t^2} A_\mu(f, t|\boldsymbol{p}) - A_\mu(\Delta f, t|\boldsymbol{p})\right)\Psi = -\frac{e}{m - \delta m(0)} (\hat{\rho}_{\mu\nu}, \hat{f})_0 (p_\nu + eA_\nu(\rho, t|\boldsymbol{p}))\Psi \tag{4.29}$$

for all Ψ in $\mathscr{F}_0^{\mathrm{EM}}$, where the time derivative is taken in the strong topology.

(2) $$\boldsymbol{A}(f, t|\boldsymbol{p}) = \mathrm{e}^{\mathrm{i}tH(\boldsymbol{p})} \boldsymbol{A}(f, 0|\boldsymbol{p})\, \mathrm{e}^{-\mathrm{i}tH(\boldsymbol{p})} \qquad \text{on } D(H(\boldsymbol{p})^{1/2}). \tag{4.30}$$

(3) (equal-time commutation relations)

$$\left[A_\mu(f, t|\boldsymbol{p}), \frac{\partial}{\partial t} A_\nu(g, t|\boldsymbol{p})\right] = \mathrm{i}(d_{\mu\nu} \hat{\bar{f}}, \hat{g})_0 \qquad \text{on } \mathscr{F}_0^{\mathrm{EM}}. \tag{4.31}$$

(4) $$W^{(n)}_{\mu_1 \ldots \mu_n}(f_1, t_1, \ldots, f_n, t_n|\boldsymbol{p}) = (\Psi_1(\boldsymbol{p}), A_{\mu_1}(f_1, t_1|\boldsymbol{p}) \ldots A_{\mu_n}(f_n, t_n|\boldsymbol{p})\Psi_1(\boldsymbol{p})). \tag{4.32}$$

Proof. Equation (4.29) follows from (3.19) and (3.20) with $K=0$. (4.30) can be proved by employing the commutation relations

$$[H(\boldsymbol{p}), a^{(r)\#}(f)] = \pm a^{(r)\#}(\omega f) \qquad r=1,2$$

where the + (respectively −) sign goes with $a^{(r)*}(\cdot)$ (respectively $a^{(r)}(\cdot)$). The commutation relations (4.31) follow from (3.13) with $K=0$. (4.32) follows from (4.5)–(4.9), lemma 4.3 and direct computations.

The above theorem shows that the quadruple $\{\mathscr{H}^{\text{ren}}(\boldsymbol{p}), \boldsymbol{A}(f, t|\boldsymbol{p}), H(\boldsymbol{p}), \Psi_1(\boldsymbol{p})\}$ is a concrete realisation of the theory of our model without infrared cut-off, but having the DES.

5. Scattering theory

In this section we consider the scattering theory of our model without the infrared cut-off given by the last section.

We first construct the asymptotic fields by the LSZ method. Let

$$a^{(r)}(f, t|\boldsymbol{p}) = \mathrm{i}\left[\frac{\partial}{\partial t} A_\mu(f^{(r)}_{t,\mu}, t|\boldsymbol{p}) - A_\mu\left(\frac{\partial}{\partial t} f^{(r)}_{t,\mu}, t\middle|\boldsymbol{p}\right)\right] \qquad f\in\mathscr{S}(\boldsymbol{R}^3) \tag{5.1}$$

where

$$f^{(r)}_{t,\mu} = (\mathrm{e}^{\mathrm{i}\omega t} e^{(r)}_\mu f/\sqrt{2\omega})^\wedge . \tag{5.2}$$

By (4.26) we have

$$a^{(r)}(f, t|\boldsymbol{p}) = \sum_{s=1}^{2} \{a^{(s)}(\mathrm{e}^{-\mathrm{i}\omega t} W_+^{(r,s)*}\, \mathrm{e}^{\mathrm{i}\omega t} f) - a^{(s)*}(\mathrm{e}^{\mathrm{i}\omega t} \bar{W}_-^{(r,s)*}\, \mathrm{e}^{\mathrm{i}\omega t} f)\} - \mathrm{i}C_\mu\left(\frac{\partial}{\partial t} f^{(r)}_{t,\mu}\middle|\boldsymbol{p}\right) \tag{5.3}$$

where

$$W_\pm^{(r,s)} = W_\pm^{(r,s)}(0). \tag{5.4}$$

Theorem 5.1. Let Ψ be in $\mathscr{F}_0^{\text{EM}}$. Then the strong limits

$$s-\lim_{t\to\pm\infty} a^{(r)\#}(f, t|\boldsymbol{p})\Psi \equiv a^{(r)\#}_{\text{out/in}}(f|\boldsymbol{p})\Psi \qquad f\in\mathscr{S}(\boldsymbol{R}^3) \tag{5.5}$$

exist and are given explicitly by

$$a^{(r)*}_{\text{in}}(f|\boldsymbol{p}) = a^{(r)\#}(f) \tag{5.6}$$

$$a^{(r)}_{\text{out}}(f|\boldsymbol{p}) = \sum_{s=1}^{2} a^{(s)}(L_0^{(s,r)} f) \tag{5.7}$$

$$a^{(r)*}_{\text{out}}(f|\boldsymbol{p}) = \sum_{s=1}^{2} a^{(s)*}(\bar{L}_0^{(s,r)} f) \tag{5.8}$$

where $L_0^{(s,r)}$ is given by (3.34) with $K=0$.

Proof Since $\bar{W}_-^{(r,s)*}$ is a Hilbert–Schmidt operator on $M_0(\boldsymbol{R}^3)$ by lemma 3.5, it follows from the lemma in the appendix that

$$\lim_{t\to\pm\infty} \|\mathrm{e}^{\mathrm{i}\omega t} \bar{W}_-^{(r,s)*}\, \mathrm{e}^{\mathrm{i}\omega t} f\|_0 = 0. \tag{5.9}$$

By the Riemann–Lebesgue lemma we also have

$$\lim_{t\to\pm\infty} C_\mu\left(\frac{\partial}{\partial t} f_{t,\mu}^{(r)}\middle| \boldsymbol{p}\right) = 0. \tag{5.10}$$

On the other hand, let

$$X(t) = \mathrm{e}^{-\mathrm{i}\omega t} W_+^{(r,s)*} \, \mathrm{e}^{\mathrm{i}\omega t} f.$$

It is easy to see that $X(t)$ is strongly differentiable with respect to t in $M_0(\boldsymbol{R}^3)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = -\frac{\mathrm{i}eQ_0 e_\mu^{(s)} \, \mathrm{e}^{-\mathrm{i}\omega t}}{2\sqrt{\omega}} \left(\frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{\omega}} e_\mu^{(r)} \hat{\rho}, f\right)_0.$$

Therefore, we get

$$X(t) = W_+^{(r,s)*} f - \tfrac{1}{2}\mathrm{i}e \int_0^t \mathrm{d}\tau \frac{Q_0 e_\mu^{(s)} \, \mathrm{e}^{-\mathrm{i}\omega\tau}}{\sqrt{\omega}} \left(\frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\sqrt{\omega}} e_\mu^{(r)} \hat{\rho}, f\right)_0.$$

By integration by parts we have

$$\left|\left(\frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\sqrt{\omega}} e_\nu^{(r)} \hat{\rho}, f\right)_0\right| < \text{constant} \frac{1}{\tau^2}$$

so that the strong limits $s-\lim_{t\to\pm\infty} X(t) \equiv X_\pm$ in $M_0(\boldsymbol{R}^3)$ exist. It is easy to check that they are given by

$$X_- = \delta_{rs} f \qquad X_+ = L_0^{(s,r)} f. \tag{5.11}$$

Since

$$s - \lim_{n\to\infty} a^{(r)}(f_n)\Psi = a^{(r)}(f)\Psi \qquad r = 1, 2 \qquad \Psi \in \mathscr{F}_0^{\mathrm{EM}}$$

as $f_n \to f$ in $M_0(\boldsymbol{R}^3)$, the desired result with respect to $a^{(r)}(f, t|\boldsymbol{p})$ follows from (5.3) and (5.9)–(5.11). In the same way we can prove the result for $a^{(r)*}(f, t|\boldsymbol{p})$.

Theorem 5.1 shows that, in our model, the LSZ asymptotic condition holds in the strong topology.

We now proceed to construct the scattering operator or S matrix. The Hilbert spaces $\mathscr{H}_{\mathrm{in/out}}(\boldsymbol{p})$ of scattering states for photons are defined by the closures of the linear span of the vectors

$$\Psi_1(\boldsymbol{p}) a_{\mathrm{in/out}}^{(r_1)*}(f_1|\boldsymbol{p}) \ldots a_{\mathrm{in/out}}^{(r_n)*}(f_n|\boldsymbol{p})\Psi_1(\boldsymbol{p}) \qquad r_j = 1, 2 \qquad f_j \in \mathscr{S}(\boldsymbol{R}^3) \qquad n \geqslant 1.$$

Lemma 5.1. (Asymptotic completeness.)

$$\mathscr{H}_{\mathrm{in}}(\boldsymbol{p}) = \mathscr{H}_{\mathrm{out}}(\boldsymbol{p}) = \mathscr{H}^{\mathrm{ren}}(\boldsymbol{p}).$$

Proof. This follows from (5.6)–(5.8) and the fact that $\mathscr{H}^{\mathrm{ren}}(\boldsymbol{p})$ is generated by

$$\{\Omega_0, a^{(r_1)*}(f_1) \ldots a^{(r_n)*}(f_n)\Omega_0 | r_j = 1, 2 \qquad f_j \in \mathscr{S}(\boldsymbol{R}^3) \qquad n \geqslant 1\}.$$

The S matrix $S(\boldsymbol{p})$ is defined as an operator from $\mathscr{H}_{\mathrm{out}}(\boldsymbol{p})$ to $\mathscr{H}_{\mathrm{in}}(\boldsymbol{p})$ by

$$S(\boldsymbol{p}) a_{\mathrm{out}}^{(r_1)*}(f_1|\boldsymbol{p}) \ldots a_{\mathrm{out}}^{(r_n)*}(f_n|\boldsymbol{p})\Psi_1(\boldsymbol{p}) = a_{\mathrm{in}}^{(r_1)*}(f_1|\boldsymbol{p}) \ldots a_{\mathrm{in}}^{(r_n)*}(f_n|\boldsymbol{p})\Psi_1(\boldsymbol{p}). \tag{5.12}$$

It follows from theorem 5.1 and lemma 5.1 that $S(\boldsymbol{p})$ is unitary. The n-photon S-matrix elements are given by

$$S^{(n)}_{r_1\ldots r_n;s_1\ldots s_n}(f_1,\ldots,f_n;g_1,\ldots,g_n|\boldsymbol{p})$$
$$=(a^{(r_1)*}_{\text{out}}(f_1|\boldsymbol{p})\ldots a^{(r_n)*}_{\text{out}}(f_n|\boldsymbol{p})\Psi_1(\boldsymbol{p}),S(\boldsymbol{p})a^{(s_1)*}_{\text{out}}(g_1|\boldsymbol{p})\ldots a^{(s_n)*}_{\text{out}}(g_n|\boldsymbol{p})\Psi_1(\boldsymbol{p})). \tag{5.13}$$

Theorem 5.2. The n-photon S-matrix elements have the form

$$S^{(n)}_{r_1\ldots r_n;s_1\ldots s_n}(f_1,\ldots,f_n;g_1,\ldots,g_n|\boldsymbol{p})$$
$$=\sum_\pi S^{(1)}_{r_1;s_{\pi(1)}}(f_1;g_{\pi(1)}|\boldsymbol{p})\ldots S^{(1)}_{r_n;s_{\pi(1)}}(f_n;g_{\pi(n)}|\boldsymbol{p}) \tag{5.14}$$

where Σ_π denotes the sum over all permutations π of $(1,\ldots,n)$. The one-photon S-matrix elements are given by

$$S^{(1)}_{r;s}(f;g|\boldsymbol{p})=\int \mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'\,S_{rs}(\boldsymbol{k}',\boldsymbol{k})\overline{f(\boldsymbol{k}')}g(\boldsymbol{k}) \tag{5.15}$$

with

$$S_{rs}(\boldsymbol{k}',\boldsymbol{k})=\delta_{rs}\delta^3(\boldsymbol{k}'-\boldsymbol{k})-2\pi\mathrm{i}\delta(|\boldsymbol{k}'|-|\boldsymbol{k}|)T(|\boldsymbol{k}|)e^{(r)}_\mu(\boldsymbol{k}')e^{(s)}_\mu(\boldsymbol{k}) \tag{5.16}$$

$$T(|\boldsymbol{k}|)=e\hat{\rho}(\boldsymbol{k})Q_0(\boldsymbol{k})/2|\boldsymbol{k}|. \tag{5.17}$$

Furthermore, the n-photon S-matrix elements are analytic with respect to e on the region Π_M. In particular, we have

$$S^{(1)}_{r;s}(f;g|\boldsymbol{p})=\delta_{rs}(f,g)_0-\sum_{n=0}^{\infty}e^{2(n+1)}(\mathrm{i}\pi/m)(\bar{F}^n\omega\hat{\rho}^2e^{(s)}_\mu[fe^{(r)}_\mu],g)_0 \qquad e\in\Pi_M. \tag{5.18}$$

Proof. Equations (5.14)–(5.17) follow from theorem 5.1 and direct computation. The second half of the theorem can be proved by using (4.20) and (5.14).

Remark. We can also show that all the off-diagonal S-matrix elements are zero. Hence the physical photon number is conserved in the scattering.

By (5.16) and (5.17) we can compute the total cross section $\sigma(\boldsymbol{k}|\boldsymbol{p})$ of the scattering of one photon with incoming momentum $\boldsymbol{k}$ by the electron with momentum $\boldsymbol{p}$. The result is

$$\sigma(\boldsymbol{k}|\boldsymbol{p})=\tfrac{8}{3}\pi r_0^2(2\pi)^6\hat{\rho}(\boldsymbol{k})^2|mQ_0(\boldsymbol{k})/e|^2 \tag{5.19}$$

where

$$r_0=e^2/4\pi m$$

is the classical electron radius.

Theorem 5.3. The total cross section of the scattering of one photon by an electron tends to that of the Thomson scattering in the low photon-energy limit if and only if the electron mass is renormalised.

Proof. We see that $\hat{\rho}(\boldsymbol{k})\to(2\pi)^{-3/2}$ as $|\boldsymbol{k}|\to 0$ and

$$\lim_{|\boldsymbol{k}|\to 0}Q_0(\boldsymbol{k})=(2\pi)^{-3/2}(e/m). \tag{5.20}$$

Therefore we get from (5.19)

$$\lim_{|\boldsymbol{k}|\to 0} \sigma(\boldsymbol{k}|\boldsymbol{p}) = \tfrac{8}{3}\pi r_0^2$$

which is the cross section of the Thomson scattering. However, this result could not be obtained if the electron mass was not renormalised, because (5.20) comes from the mass renormalisation. Thus the theorem follows.

6. Conclusion

We have considered the model defined by the Hamiltonian (2.8) or (2.14) with (respectively without) infrared cut-off and analysed the spectral properties (corollaries 3.2 and 3.3), establishing the existence (respectively absence) of the DES (theorems 3.3 and 3.4 and corollary 3.1). After the reconstruction of the theory without infrared cut-off, but having the DES, by means of the Wightman distributions, we have developed the scattering theory of photons by an electron and shown that the mass renormalisation of the electron is necessary to obtain the cross section of the Thomson scattering in the low photon-energy limit (theorem 5.3).

An interesting and important problem now is to extend our results to the case without the dipole approximation. The model in this case is of course not explicitly soluble. However, the proof of existence (respectively absence) of the DES of the Hamiltonian with (respectively without) infrared cut-off and the construction of the scattering theory would go in parallel with Fröhlich (1973). In this abstract approach, however, we shall have to devise some appropriate calculation scheme to study the behaviour of the scattering cross section, e.g., its low photon-energy limit.

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Appendix

Lemma. Let S be a Hilbert–Schmidt operator on $L^2(R^n)$, $n \geqslant 1$. Then,

$$\lim_{t\to\pm\infty} \|S\,\mathrm{e}^{\mathrm{i}Rt}f\|_{L^2(R^n)} = 0 \qquad f \in L^2(R^n)$$

where

$$R(x) = |x| \qquad x \in R^n.$$

Proof. We can write

$$(S\,\mathrm{e}^{\mathrm{i}Rt}f)(x) = \int \mathrm{d}y\, S(x, y)\,\mathrm{e}^{\mathrm{i}|y|t}f(y)$$

where $S(\cdot,\cdot) \in L^2(R^{2n})$ is the Hilbert–Schmidt kernel of S. The square integrability of $S(\cdot,\cdot)$ and the Riemann–Lebesgue lemma imply that

$$\lim_{t\to\pm\infty} (S\, e^{iRt}f)(x) = 0 \qquad \text{AE}\,x.$$

Furthermore, we have by the Schwarz inequality

$$|(S\, e^{iRt}f)(x)|^2 \leq \|f\|^2_{L^2(R^n)} \int dy\, |S(x,y)|^2 \qquad \text{AE}\,x.$$

The RHS does not depend on t and is integrable with respect to dx. Thus, by the dominated convergence theorem, we get the desired result.

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